A Physical Point Lattice Calculation of the $K \to \pi\ell^+\ell^-$ Decay

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Introduction

The rare kaon decay $\boldsymbol{K \to \pi\ell^+\ell^-}(\ell = e, \mu)$ is a flavor changing neutral current (FCNC) process, which is forbidden a tree level in the Standard Model (SM). This process is thus a channel with which to to look for signs of new physics, in particular it is suitable to look for possible lepton flavor universality violation in the kaon sector [1].

There are ongoing experiments at NA62 [2] and prospective experiments at LHCb [3] that will give insight into the K^+ and K_S decays respectively. Improving theoretical understanding in tandem with experiment is necessary in order to rigorously test the SM.

As long distance effects dominate the $K \to \pi\ell^+\ell^-$ decay perturbative methods are not suitable for a full theoretical investigation. Instead we use lattice quantum chromodynamics (LQCD), which discretizes the theory of QCD on a finite hypercubic lattice, to make non-perturbative predictions.

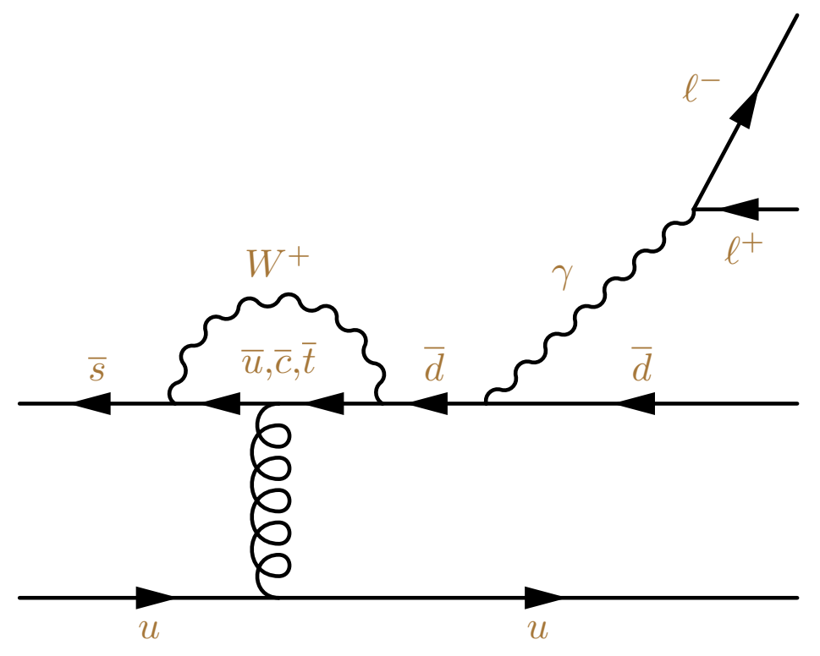


Figure 1: Feynman diagram representing the $K \to \pi\ell^+\ell^-$ decay

Long Distance Amplitude

The long distance amplitude we wish to compute is given by

$$\mathcal{A}^j_\mu(q^2) = \int \mathrm{d}^4x \left\langle \pi^j(\mathbf{p}) \middle| \mathrm{T}\left[J_\mu(0)H_W(x)\right] \middle| K^j(\mathbf{k}) \right\rangle$$

where J_μ is the electromagnetic current, H_W is an effective weak Hamiltonian, $q = k - p$, and $j = +, 0$. This can be written in terms of an electromagentic form factor, $V_j(z)$, as

$$\mathcal{A}^j_\mu(q^2) = \frac{-iG_F}{(4\pi)^2}\left(q^2(k+p)_\mu - q_\mu\left(M_K^2 - M_\pi^2\right)\right)V_j(z),$$

where $z = q^2/M_K^2$. This form factor can be parameterized as

$$V_j(z) = a_j + b_j z + V_j^{\pi\pi}(z)$$

where a and b are constants, that have been found only through experimental prediction, and $V_j^{\pi\pi}(z)$ is a known quantity introduced to account for $\gamma^* \to \pi\pi$ effects. **The aim of this project is to perform an ab initio calculation of a and b using LQCD with physical quark masses.**

e	$\lvert a_S\rvert = 1.06^{+0.26}_{-0.21}$	$a_+ = -0.578(16)$	$b_+ = -0.779(66)$
μ	$\lvert a_S\rvert = 1.54^{+0.40}_{-0.32}$	$a_+ = -0.575(39)$	$b_+ = -0.813(145)$

Table 1: Phenomenological predictions for a and b [4]. The sign of a_S is not know and there are no conclusive predictions for b_S.

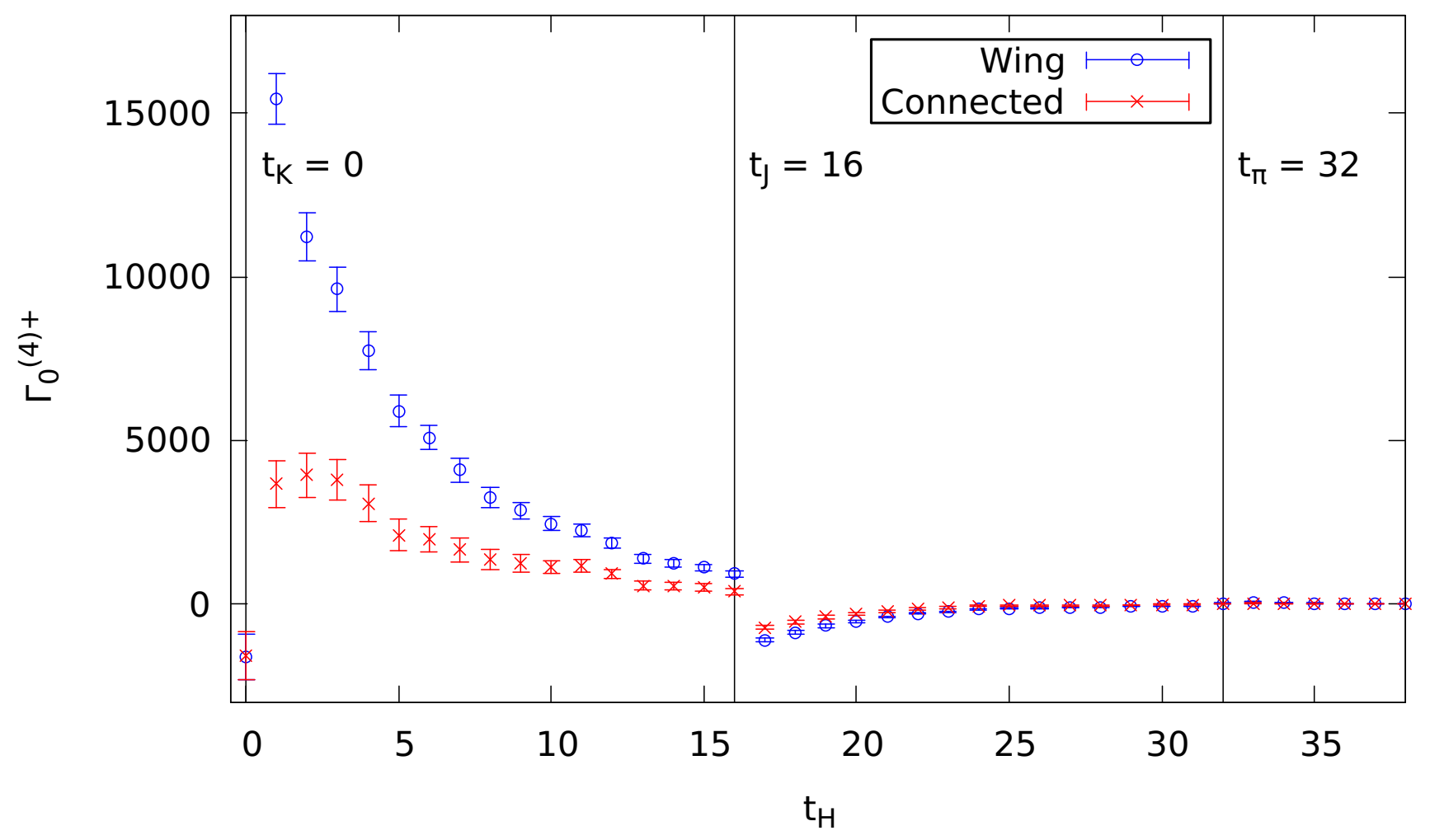
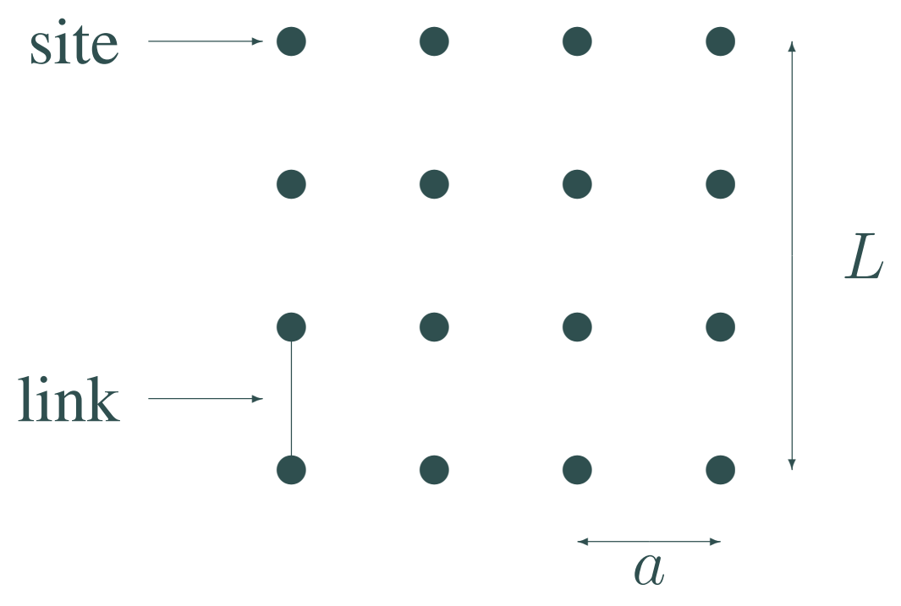


Figure 2: Contibutions of the “Non-Eye” diagrams to $\Gamma_0^{(4)+}$

Lattice QCD



LQCD seeks to discretize spacetime on a finite hypercubic lattice where fermion fields are defined on sites, separated by lattice spacing a, and gauge fields are defined by the links between neighboring sites.

By moving to **Euclidean spacetime** the QCD path integral:

$$\langle 0|\, \mathcal{O}_1 \ldots \mathcal{O}_n \,|0\rangle = \frac{1}{Z}\int \mathcal{D}\left[A,\psi,\bar{\psi}\right] \mathcal{O}_1 \ldots \mathcal{O}_n e^{-S_E[A,\psi,\bar{\psi}]}$$

can be approximated as a Monte Carlo simulation. Here S_E is the Euclidean QCD action and Z is the partition function:

$$Z = \int \mathcal{D}\left[A,\psi,\bar{\psi}\right] e^{-S_E[A,\psi,\bar{\psi}]}$$

Software & Machine

- The physics code base **Grid** lays the foundation for simulations.

Grid is a C++ mathematical object library [5] designed to deliver a high level data parallel approach that targets multiple types of parallelism, such as MPI, OpenMP and short vector parallelism, and multiple computer architectures, including Skylake CPUs, KNL CPUs, and NVIDIA GPUs.

- The measurement workflow management system **Hadrons** sets up the structure of the runs.

This Grid-powered C++ framework is based on modular dataflow programming, where each measurement step (I/O, inversions, contractions, etc.) is implemented as an individual module. This modularity is coupled with flexible I/O & control, and a global virtual machine that manages scheduling and garbage collection.

- The runs are performed on the DiRAC Extreme Scaling system.

Known as **Tesseract**, this is a 1468-node HPC system that is housed at EPCC’s Advanced Computing facility. Each node consists of two 2.1 GHz, 12-core Intel Xeon (Skylake) Silver 4116 processors and 96 GB of memory. In addition, there are 8 GPU compute nodes each with 4 NVidia V100 (Volta) GPU accelerators. All nodes are connected by a single Intel Omni-Path Architechture fabric and can access a 3 PB Lustre file system.

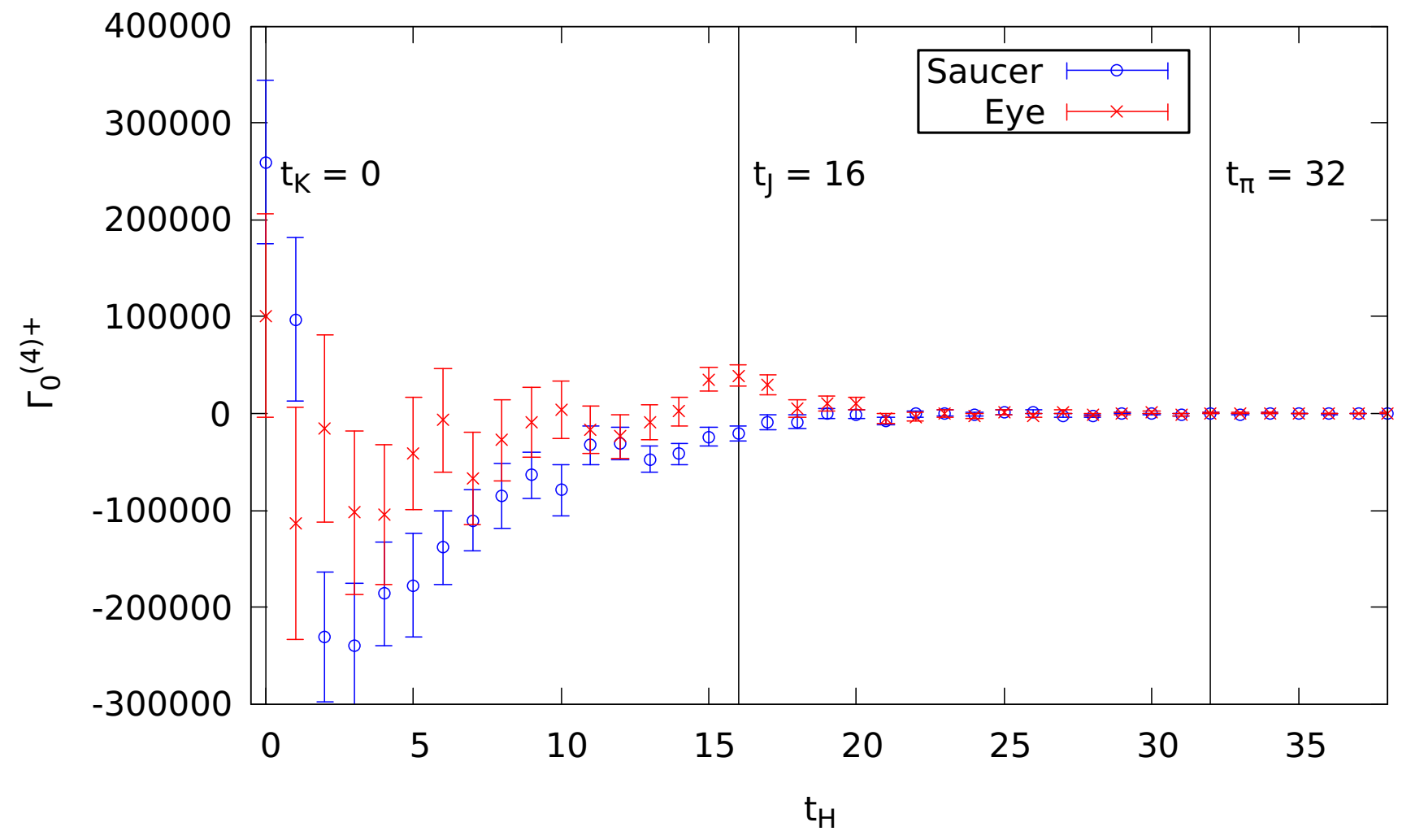


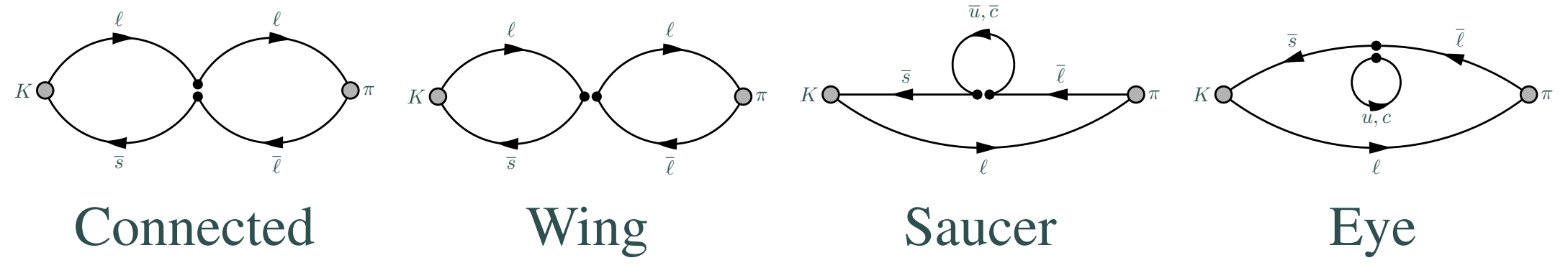
Figure 3: Contibutions of the light quark “Eye” diagrams to $\Gamma_0^{(4)+}$

Extracting the Decay Amplitude

In order to computed the long distance amplitude the unintegrated 4-point correlation function is defined as follows:

$$\Gamma^{(4)j}_\mu\left(t_H, t_J, \mathbf{k}, \mathbf{p}\right) = \int d^3\mathbf{x} \int d^3\mathbf{y}\; e^{-i\mathbf{q}\cdot\mathbf{x}} \left\langle \phi_\pi\left(t_{\pi^j}, \mathbf{p}\right) T\left[J_\mu\left(t_J, \mathbf{x}\right) H_W\left(t_H, \mathbf{y}\right)\right] \phi^\dagger_{K^j}\left(t_K, \mathbf{k}\right) \right\rangle$$

Performing the quark Wick contraction in the correlation function generates the Feynman diagrams which need to be evaluated. In the case of isospin symmetry in the quark masses, $m_u = m_d$, 20 diagrams need to be computed for the charged correlator, which can be split into 4 classes as shown in Figure (4).
Each class has 6 sub-diagrams with current insertions, an example of which can be seen in Figure (5).



Connected
Wing
Saucer
Eye

Figure 4: The diagrams contributing to the Wick contractions for $K \to \pi\ H_W$ 3-pt functions.

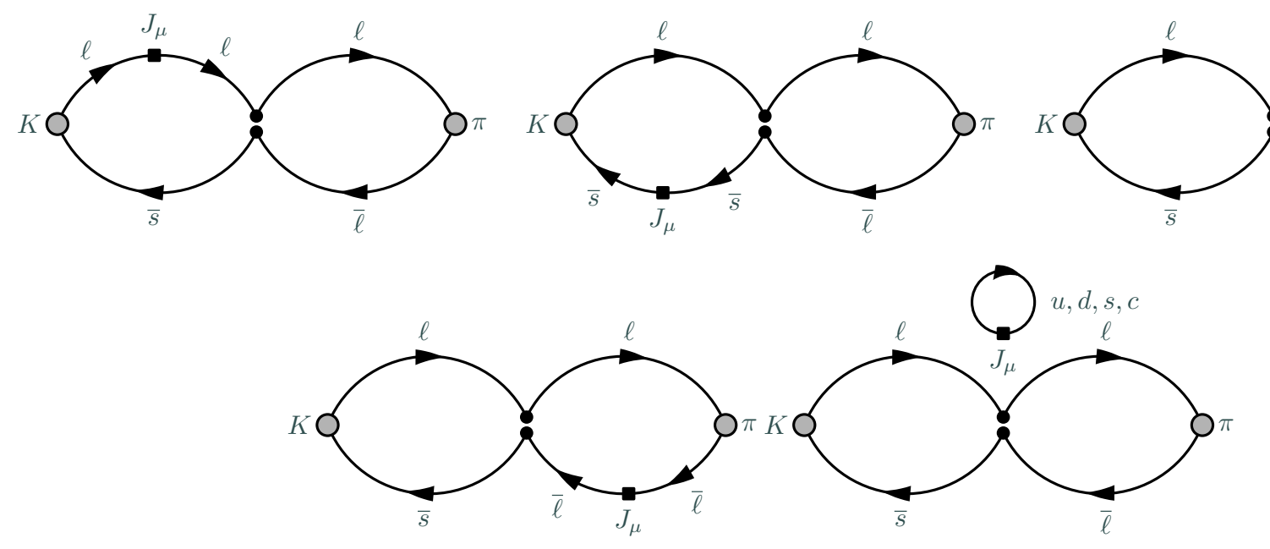


Figure 5: The “Connected” diagrams needed to calculate the rare kaon decay correlator, with current insertions.

After dividing out the source/sink and normalization factors the “reduced” correlator $\tilde{\Gamma}^{(4)j}_\mu$ is integrated over t_H and t_J to give:

$$I^{(4)j}_\mu\left(T_a, T_b, \mathbf{k}, \mathbf{p}\right) = e^{-(E_\pi(\mathbf{p}) - E_K(\mathbf{k}))t_J} \int_{t_J - T_a}^{t_J + T_b} dt_H\; \tilde{\Gamma}^{(4)}_\mu\left(t_H, t_J, \mathbf{k}, \mathbf{p}\right)$$

Finally we can write the Minkowski long distance amplitude in relation to the integrated 4-point correlator with the exponentially growing contributions subtracted, $\tilde{I}^{(4)j}_\mu$

$$\mathcal{A}^j_\mu\left(q^2\right) = -i\frac{G_F}{\sqrt{2}} V^*_{us} V_{ud} \lim_{T_a, T_b \to \infty} \tilde{I}^{(4)j}_\mu\left(T_a, T_b, \mathbf{k}, \mathbf{p}\right),$$

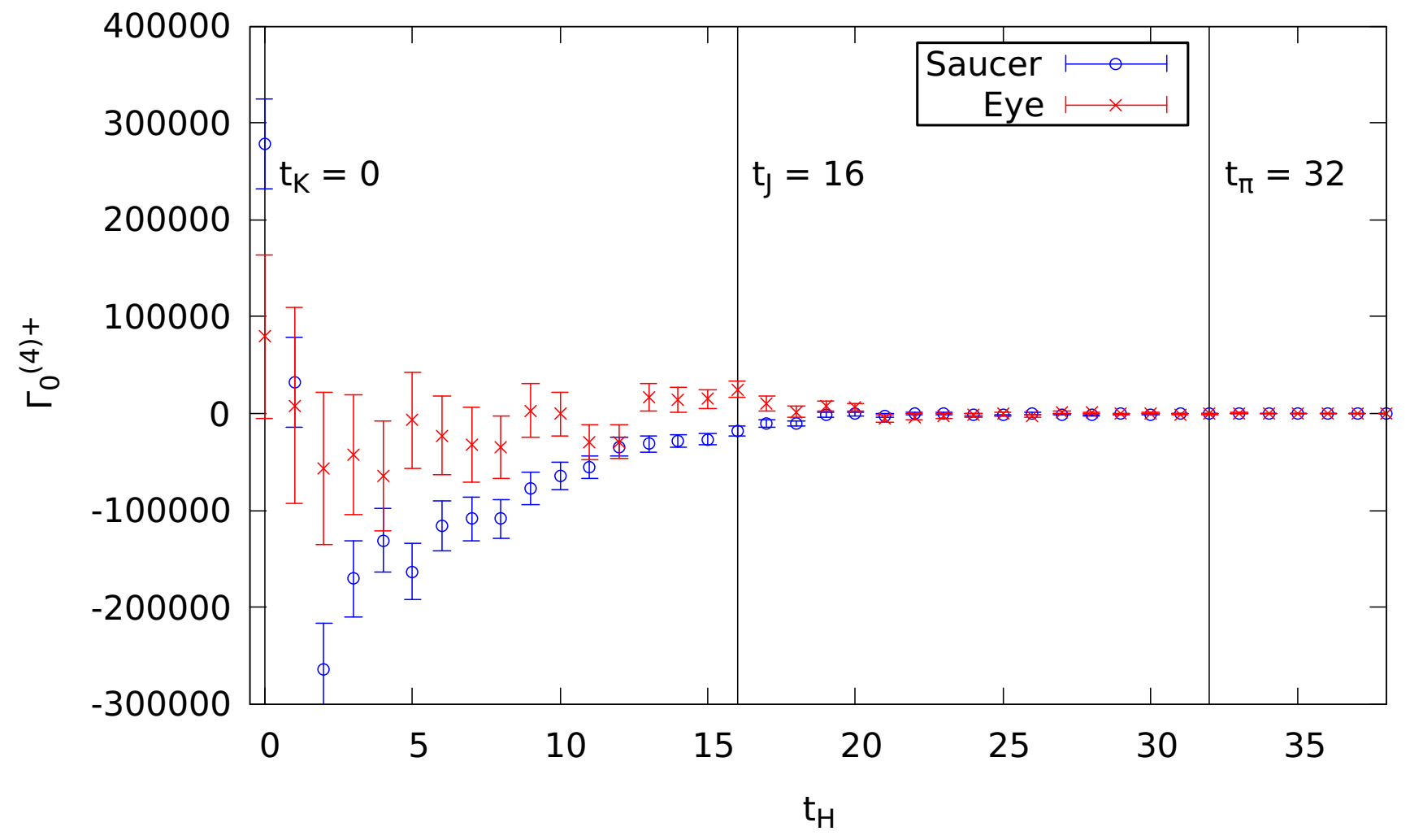


Figure 6: Contibutions of the charm quark “Eye” diagrams to $\Gamma_0^{(4)+}$

Physical Point Calculations

The calculations are performed on a $48^3 \times 96$ Domain Wall Fermion (DWF) gauge configuration [6] with a lattice spacing of $a^{-1} = 1.73$ GeV and meson masses of $\mathrm{M}_\pi \approx$ **140 MeV,** and $\mathrm{M}_\mathbf{K} \approx$ **500 MeV** with $2 + 1$ dynamical quark flavors.
We first look at the K^+ decay where the kaon is simulated at rest and the pion is given a final state momentum of $\mathbf{p}_\pi = \frac{2\pi}{L}(1,0,0)$, with $\mu = 0$.
Preliminary results for the “Non-Eye” (Connected and Wing) diagrams can be seen in Figure (2) and the light and charm “Eye” (Saucer and Eye) diagrams can be seen in Figures (3) and (6). These are the results of a bootstrap re-sampling of calculations from 16 gauge configurations, each with 6 time-translational hits. The loops have been computed with 2^4 sparsened noises.

Future Outlook

- Variance reduction methods are being investigated to improve the signal on the “Eye” diagrams.
- Disconnected diagrams need to be considered.
- Three lighter-than-physical charm quark masses have been used to compute the charm quark loops, extrapolation to the physical mass is required.
- The initial kinematics of the final state pion give a value of $z = 0.001$, which will give a good indication of what a_+ is.
- Calculations with $\mathbf{p}_\pi = \frac{2\pi}{L}(1,1,0)$ and $\frac{2\pi}{L}(1,1,1)$ are to follow, allowing us to perform a linear fit to calculate a_+ and b_+

References

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