QED Corrections to Hadronic Amplitudes

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Abstract

Here I will present progress from the RBC-UKQCD collaboration on obtaining $\mathcal{O}(\alpha)$ corrections to matrix elements. I will introduce how QED is implemented on the lattice and how we may extract the corrections from Euclidean correlators. Preliminary results will be presented and future plans are discussed.

Introduction

In Nature, isospin symmetry is broken slightly and this may be observed in the neutron-proton mass difference. Though the difference is small, it is exactly this per-mille effect that determines the abundance of heavy nuclei via BBN, which in turn shapes the Universe at late times. This breaking of isospin symmetry is due to

1. the different masses of the up and down quarks and
2. the up and down quarks are electrically charged.

Up until recently, predictions from Lattice QCD mostly neglect isospin-breaking effects since $\frac{m_u - m_d}{\Lambda_{\text{QCD}}} \lesssim 0.01$ and $\alpha = \frac{e^2}{4\pi} \sim 0.0073$, *i.e.* these percent-level corrections are negligible so long as statistical errors are greater than 1%. Thanks to continual improvements in the high performance computing front, lattice simulations in the recent decade can be performed in sufficiently large volumes such that these effects may be accounted for.

Lattice QCD(+QED) in a Nutshell

Consider the vacuum expectation value of an observable O in the (Euclidean) path integral formalism. Including both QCD and QED, it is

$$\langle O \rangle = \frac{1}{Z}\int D\bar{\psi} D\psi DU DA\, O[\psi, \bar{\psi}, U, A] e^{-S_F[\psi,\bar{\psi},U,A] - S_g[U] - S_\gamma[A]}, \tag{1}$$

with

$$\begin{aligned} S_F &= a^4 \sum_{x,y\in\Lambda} \bar{\psi}(x) D(x|y)\psi(y), \\ \text{and} \quad S_g &= \frac{1}{g^2}\sum_{x\in\Lambda}\sum_{\mu<\nu} c_0 \mathrm{Re}\,\mathrm{Tr}\left[\mathbb{1} - U_{\mu\nu}(x)\right] \\ &\quad + c_1 \mathrm{Re}\,\mathrm{Tr}\left[\mathbb{1} - R_{\mu\nu}(x)\right], \end{aligned} \tag{2}$$

as the fermionic and Iwasaki gauge action[1] respectively and $S_\gamma[A] = -\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu}$ is the U(1) photon action. Here, $D(x|y)$ is the discretised version of the Dirac operator, $U_{\mu\nu}(x)$ is a plaquette, $R_{\mu\nu}$ is the 2×1 Wilson loop (a.k.a a rectangle), c_0, c_1 are the corresponding weights and A is the photon field.

In this work, $D(x|y) \to D^{\text{DW}}(x, s|y, r)$ is the domain wall Dirac operator, where the indices s, r labels the extent of the fifth dimension. Similarly, the fermion fields are promoted to 5-dimensional domain wall fermions[2]. We are also working in an electro-quenched approximation such that the fermion determinant does not depend on the photon field, A. Thus, the electromagnetic field plays the role of a Gaussian weighting, $e^{-S_\gamma[A]}$.

The Photon on a Lattice

In this work, we use the QED$_{\text{L}}$ formulation. Here, the photon's spatial zero mode is subtracted off every timeslice[3]. In momentum space and using Feynman gauge ($\xi = 0$), the photon action is

$$S_\gamma[A] = \frac{1}{2TL^3}\sum_{k^0, \vec{k}\neq\vec{0}}\sum_{\mu=1}^{4} \hat{k}^2 |\tilde{A}_\mu(k)|^2, \tag{3}$$

where $\hat{k}^\mu = \frac{2}{a}\sin\frac{ak_\mu}{2}$. This allows us to preserve gauge invariance in a finite, periodic lattice. However, this method of removing zero modes introduces an interaction term that is non-local in space. It can be shown that such non-local terms do not contribute in the infinite volume limit when we match with physical results (see [3]).

Extracting QED Corrections

We are interested in the QED corrections at $\mathcal{O}(\alpha)$ to the decay channels $\pi^+/K^+ \to l^+\nu$ and their charge conjugate modes. These processes give us access to CKM matrix elements such as V_{ud} and V_{us}. Obtaining a full QCD+QED lattice prediction of these matrix elements will give us better theoretical control of our uncertainties when comparing with phenomenological data (*e.g.* see [4]).

At order $\mathcal{O}(\alpha)$, there are six connected diagrams that involve a single photon. Figure 1(c-f) are diagrams where the initial/final state receives QED corrections - these are called **factorisable** diagrams. On the other hand, Figure 1(a,b) are **non-factorisable** diagrams, where the photon couples the initial and final state. In the following analysis we will focus on the latter.

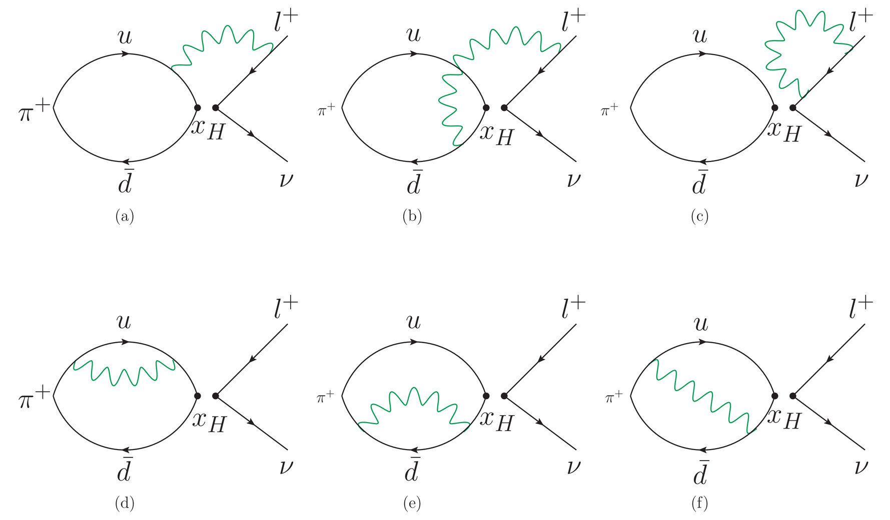

Figure 1: The $\mathcal{O}(\alpha)$ diagrams for one of the P_{l2} decay channels. The diagrams can be categorised into non-factorisable (a-b) and factorisable (c-f) contributions. The case for kaons is identical.

Recall that the amplitude of this decay has the following form:

$$\begin{aligned} \mathcal{M}^{rs} &= \left\langle l, r; \nu, s \middle| \bar{\nu}\tilde{H}_W \middle| P^+ \right\rangle, \\ &= \bar{u}^s_\alpha(p_\nu)\mathcal{M}_{\alpha\beta} v^r_\beta(p_l), \\ \text{with} \quad \mathcal{M}_{\alpha\beta} &= \Gamma^4_{\alpha\beta} M_P f_P + \delta\mathcal{M}_{\alpha\beta} + \mathcal{O}(\alpha^2), \end{aligned} \tag{4}$$

where M_P and f_P are the mass and decay constant of the pseudoscalar, $\Gamma^4 = \gamma^4(1-\gamma^5)$ and $\delta\mathcal{M}$ is the $\mathcal{O}(\alpha)$ correction to the amplitude.

Omitting kinematic factors from phase space integration, we express the decay width as

$$\begin{aligned} \Gamma(P^+ \to l^+\nu) \propto \sum_{r,s} |\mathcal{M}^{rs}|^2 &= \Gamma_0 + \delta\Gamma, \\ &= \Gamma_0\left(1 + \delta R_P + \mathcal{O}(\alpha^2)\right), \end{aligned} \tag{5}$$

where Γ_0 is the tree-level decay width and δR_P represents the linear order QED corrections. We can build the decay widths from correlators of the following form:

$$C(dt_l, dt_P)_{\alpha\beta} = \sum_{\vec{x}_l, \vec{x}_H, \vec{x}_P} \langle 0 | \bar{l}_\alpha(x_l) \tilde{H}_{W\beta}(x_H) \phi^\dagger(x_P) | 0 \rangle\, e^{i\vec{p}_\nu \vec{x}_H}, \tag{6}$$

where $dt_l = t_l - t_H$ and $dt_P = t_H - t_P$ are the source-sink separations of the lepton and pseudoscalar meson respectively. $\bar{l}$ is the anti-lepton annihilation operator, ϕ is the pion creation operator and the phase factor ensures conservation of 3-momentum. $\tilde{H}_W$ is the reduced weak Hamiltonian with the neutrino operator amputated. Finally, we saturate the spinors α and β by including the neutrino propagator analytically to obtain

$$\Gamma_0 \propto \mathrm{Tr}\left[-\not{p}_{\nu,E}\Gamma^4(-\not{p}_{l,E} + im_l)\Gamma^4\right] |f_P|^2, \tag{7}$$

$$\delta R_P \propto \frac{\mathrm{Tr}\left[\not{p}_{\nu,E}\delta C(dt_l, dt_P)\Gamma^4\right]}{\mathrm{Tr}\left[\not{p}_{\nu,E} C_0(dt_l, dt_P)\Gamma^4\right]}, \tag{8}$$

where $(-\not{p}_{l,E} + im_l)$ and $\not{p}_{\nu,E}$ are the Euclidean space lepton and neutrino propagator respectively, C_0 is the tree-level correlator and δC is the $\mathcal{O}(\alpha)$ contribution to Equation 6. For a fixed dt_l, we should expect a plateau at large dt_P.

Results

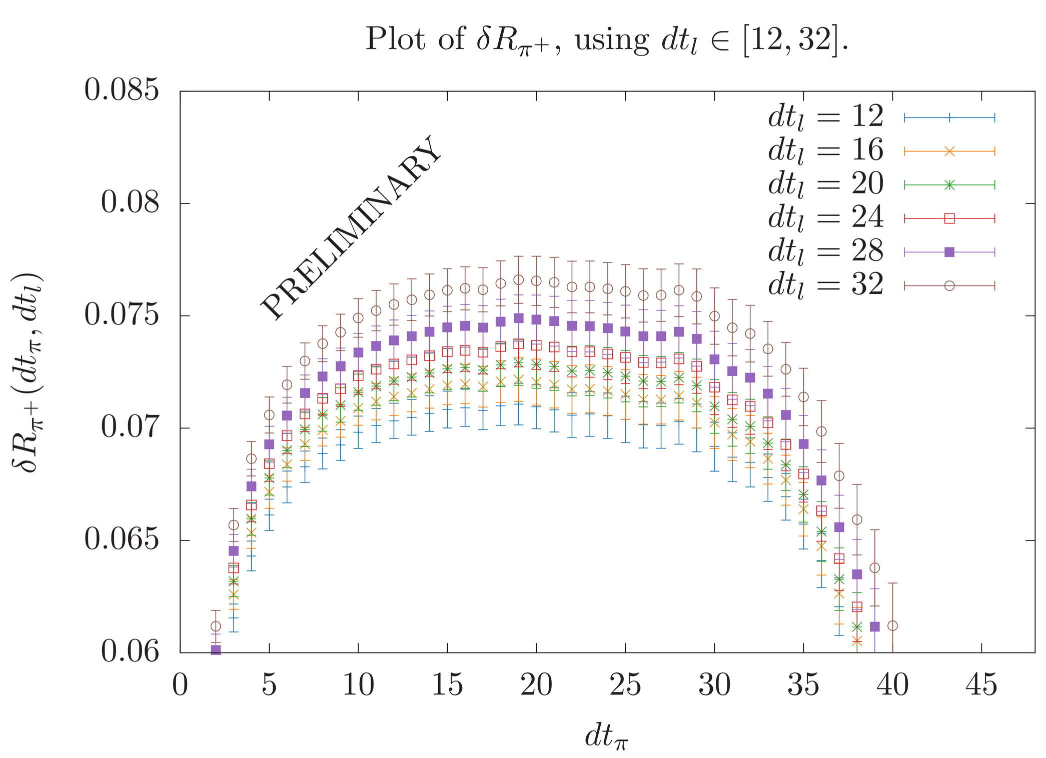

Figure 2: $\delta R_{\pi^+}(dt_\pi, dt_l)$ for the pion in the first half of the lattice, $0 \leq t_H \leq 48$.

Figure 2 is the latest lattice calculation of the non-factorisable δR_{π^+} for a charged pion. We have used physical point Möbius Domain Wall Fermions, with $T \times L^3 \times L_s = 96 \times 48^3 \times 24$, lattice spacing $a \simeq 0.12$ fm and $N_f = 2+1$. The data for all diagrams needed in δR_{π^+} and δR_{K^+} are also available.

In this plot, δR_{π^+} is generated for a range of dt_l. Here, we have averaged over all 96 different t_π and performed bootstrap resampling over 44 configurations. From the figure, we see the effects of excited state and backward contribution for different dt_l. The analytic form of these effects are known and this can be accounted for in a simultaneous fit across all dt_l data.

Conclusions

- Thanks to improved availability of computing resources, percent-level corrections due to isospin-breaking effects are realisable on the lattice.
- Accounting for QED effects in our lattice prediction of CKM matrix elements will be a crucial milestone in the age of precision physics.

Outlook

- Extract the QED corrections to V_{us}/V_{ud} by constructing and fitting $\delta R_{K^+}/\delta R_{\pi^+}$.
- Renormalise $\tilde{H}_W$ to extract $\mathcal{O}(\alpha)$ corrections to V_{ud} and V_{us} independently.
- Apply strategy to study phenomenologically interesting channels, such as $K^+ \to \pi^0 l^+\nu$, $K^0 \to \pi^\pm l^\mp \nu$ and their charge conjugate modes.

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